Rational and pragmatic assessment of uncertain deduction

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To what use can probabilistic deductive schemas be put?

As is well known, for each of the basic deductive schemas that are used for modelling individuals' reasoning (typically in private deliberation or argumentation), knowing the probability of the premises it is possible to determine the probability of the conclusion, either as a point value, or most often as a coherence interval. A point value has the practical advantage that it clearly indicates the correct assessment of the conclusion that is to be expected. A coherence interval indicates the bounds within which the belief in the conclusion must lie to be rationally justified, that is, to follow the rules of the probability calculus, given the individuals' degree of belief in the premises.

Example: *and-introduction:* A; C ∴ A&C

Given degrees of belief p(A) for the premise A, p(C) for the premise C, the probability of the conclusion p(A&C) must belong to this interval:

$p(A\&C) \in [\max\{0, p(A)+p(C)-1\}, \min\{p(A), p(C)\}]$

However, there is something paradoxical here because the wider the interval, the greater the reasoners' «freedom» of assessment, to such an extent that any assessment is permitted when the coherence interval reaches [0, 1]. So, at first sight, many arguments seem to be of little or no practical interest, whether it is to make a decision or to argue. For instance, one may think it <u>likely</u> that:

if Mary is at home, the TV is on (if A then C),

and one is licensed to infer that:

it is very <u>likely</u> that if the TV is off, Mary is out (if not-C then not-A),

as well as that:

it is very <u>unlikely</u> that if the TV is off, Mary is out.

On an intuitive basis, the former conclusion may be more readily expected and accepted, but the latter is not surprising if it is known that *most of the time Mary is at home.*

This suggests that additional information may play a crucial role, possibly difficult to trace and control.

I will consider the following question:

Are there any circumstances that render schemas with wide intervals useful?

and the answer that:

There are pragmatic reasons that may justify their use.

Before developing the answer, it will be useful to identify a few properties that an inference schema may or may not possess.

A few properties that an inference schema may or may not possess

• There are schemas such that the probability of the conclusion cannot be lower than that of the premise (or, if there are several premises, lower than the probability of at least one premise). I will call them «conservative° ».

Example: *and-elimination*

A&C $\therefore$ A **p(A) $\in$ [p(A&C), 1]**

• The opposite may be the case: the probability of the conclusion cannot be higher than that of the premise. I will call them «dissipative ».

Example: *or* to *if-not*

A or C $\therefore$ If not-A then C p(If not-A then C) $\in$ [0, p(A or C)]

° Adams' (1996) minimum probability preservation

• Another interesting property is that for some schemas a high degree of belief in the premise(s) guarantees a high (or nearly as high) degree of belief in the conclusion. These schemas can be called «robust° ».

Example. *Modus Ponens:*

if A then C; A $\therefore$ C

$$p(C) \in [\, p(\text{if A then C}) \times p(A),\ p(\text{if A then C}) \times p(A) + 1 - p(A) \,]$$

If Mary declares that she is *very* confident that

Peter will finish his homework (C),

a reasonable (coherent) justification could be:

"because I am *highly* confident that *if Peter works the whole afternoon he will finish,* and also I am *highly* confident that *he will work the whole afternoon*".

In a similar vein, the confidence intervals can be exploited to decide or examine the extent to which a conclusion is assertable, given the reasoner's degree of belief in the premises, taking "more likely than not" as a pragmatic/argumentative criterion of assertability.

° Adams's (1996) high probability preservation

Taking "more likely than not" as a pragmatic/argumentative criterion of assertability:

This use of the coherence interval differs from the mere endorsement of a conclusion that falls within the interval limits. For instance:
suppose that p(conclusion) equals, say, 1/3 and falls in the interval: this means that the reasoner is coherent, but:
it may be important <u>to know whether there are additional conditions that would justify an assessment of p(conclusion) ≥ 1/2.</u>

I will distinguish the case where the lower bound is a function of the probability of the premises from the case where the lower bound is null.

Example for the first case:
and-introduction: A, C $\therefore$ A&C

$$\mathbf{p(A\&C) \in [\max\{0, p(A)+p(C)-1\}, \min\{p(A), p(C)\}]}$$

For A&C to be assertable it suffices that $p(A) + p(C) - 1 \geq 1/2$, that is, $p(A) + p(C) \geq 3/2$.

Remarkably, it is often possible to treat the question using non mumerical values. Here, a strong belief in A and in C warrants the assertion of A&C; but a moderate degree of belief in each premise does not warrant even a moderate degree of belief in A&C.

Case where the lower bound is a function of the probability of the premises (cont'd).

Another example.

Proof by cases: P1: if A then C; P2: if not-A then C $\therefore$ C

p(C) $\in$ [min {P1, P2}, max {P1, P2}]

It results that if each premise is assertable, so is C. Moreover, a reasoner can have high confidence in C on the grounds that she has high confidence in the least believable premise. Again this does not require numerical calculations.

In brief, when the lower bound is a function of the probability of the premises, it is possible to determine a condition on the probability of the premises that guarantees the assertability of the conclusion. The reasoners are justified in asserting C by their degree of belief in P (in the sense defined). The interval is exploited not relatively to its span but to its lower bound.

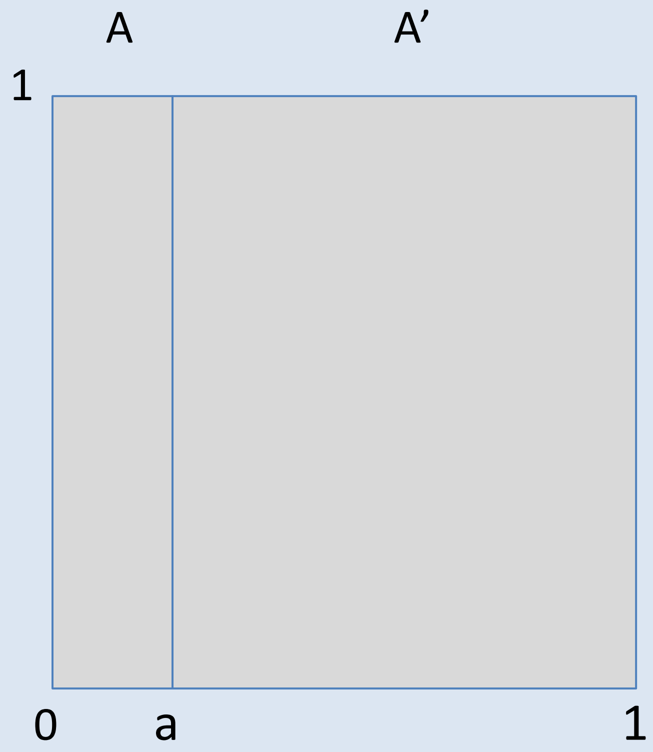
In the second case, the one where the lower bound is null, the probability of the conclusion may be null. (In terms of p-validity, the inference is not p-valid). The question of the usefulness or reasonableness of the schema arises: Are there conditions (that is, additional assumptions such as the degree of belief in another proposition) under which a high degree of belief in the conclusion is justified? Remarkably, this includes the non informative case where the interval is [0, 1].

In what follows I will :
- consider the question of the assertion of a conclusion when the schema is lower bounded by zero, and especially when it is probabilistically non informative.
- choose a few well-known schemas and analyse them to identify pragmatically-based assumptions that make the schemas intuitively attractive.

I will do this using a diagrammatic representation of de Finetti’s coherence approach that allows the derivation of the coherence intervals and highlights the reasons why the intervals are bounded by zero.

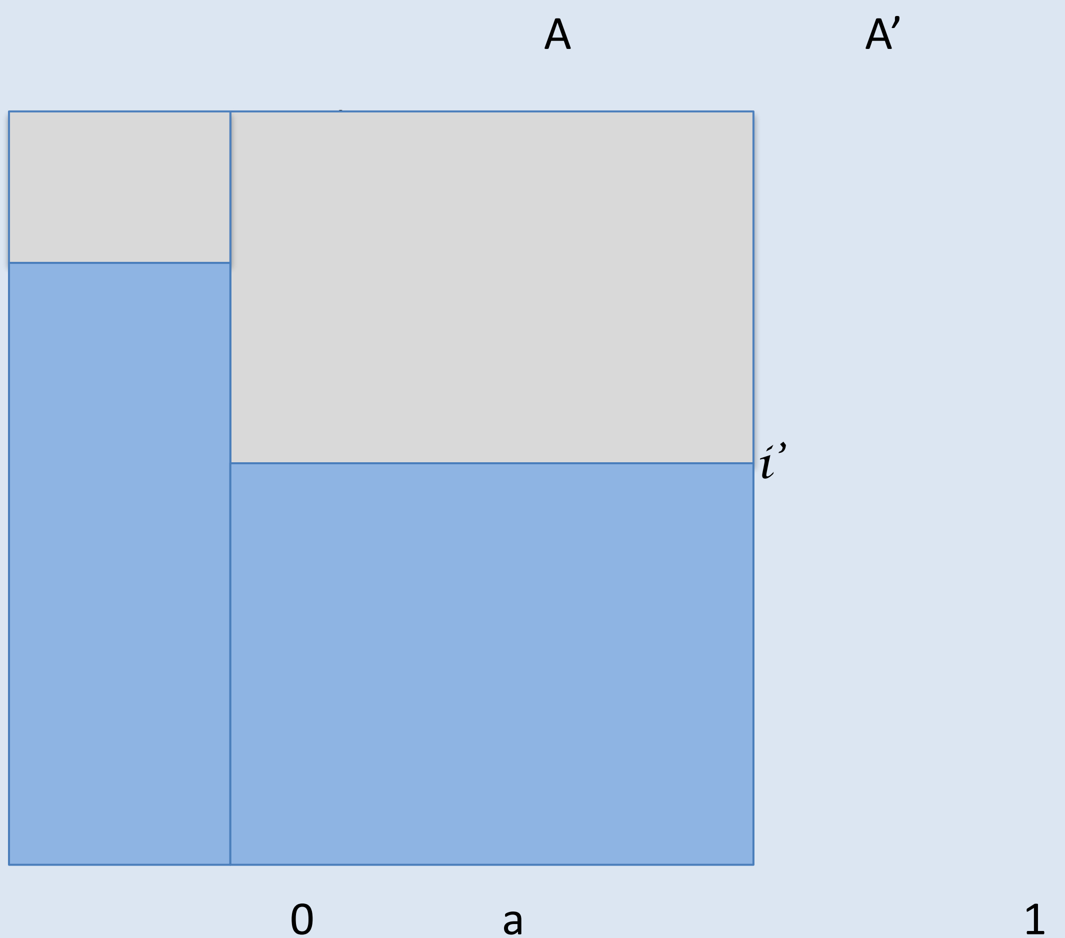
A physical implementation of de Finetti's probability theory: a water tank analogy

A two-compartment cubic water tank



$0 \leq a \leq 1$

Filling the tank



$0 \leq a \leq 1$

$0 \leq i \leq 1$

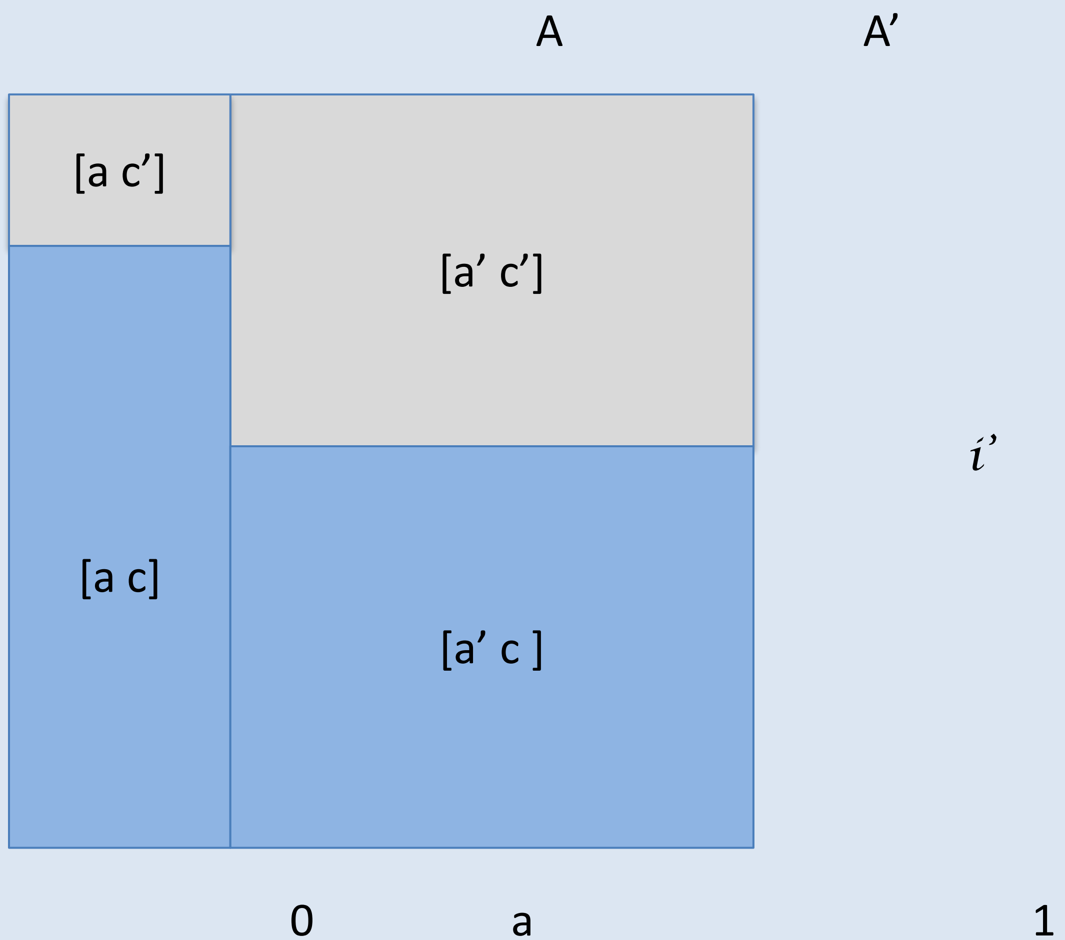
$0 \leq i' \leq 1$

$0 \leq c \leq 1$

i, *i’* = levels

c = overall content

Measuring the content (and the void)

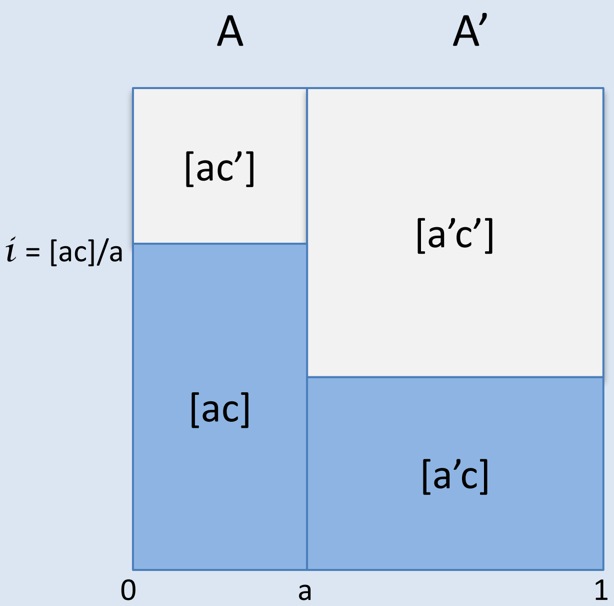


$[ac] = a \times i$ $\quad$ $[a'c] = (1- a) \times i'$

$[ac'] = a \times (1- i)$ $\quad$ $[a'c'] = (1- a) \times (1- i')$

$i = [ac] / a$ $\quad$ $i' = [a'c] / 1- a$

The probabilistic interpretation



a = P(A) **c = P(C)**
i = P(C|A)

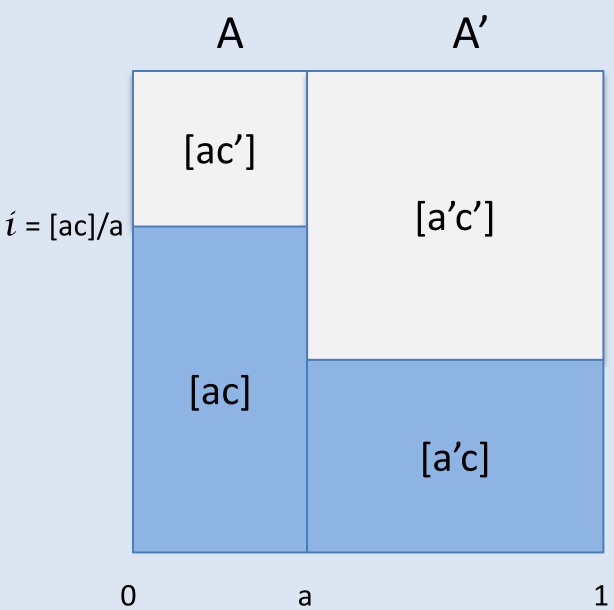
- A is the event "the left compartment occupies the whole tank";
a, the capacity of A, is the degree to which the whole tank is occupied by A: a = P(A)

- C is the event "the tank is filled up with liquid";
c (the volume of liquid in the tank) is the degree to which the tank is filled with liquid: c = P(C)

- The event *restricted* to the compartment A, "A is filled up with liquid" is a *conditional event* denoted by C/A; i
(the level of liquid in A) is the degree to which A is filled with liquid: i = P(C|A), which is the probability of *if A then C.*

- Similarly, the event *restricted* to the liquid "all the liquid is contained in A", is a conditional event denoted by A/C;
its probability denoted by P(A|C) is the degree to which all of the liquid is in A, which is the probability that *if C then A.*

The probabilistic interpretation



$a = P(A)$ **$c = P(C)$**

$i = P(C \mid A)$

$[ac] = P(A \text{ AND } C) = a \times i = P(A) \times P(C \mid A)$

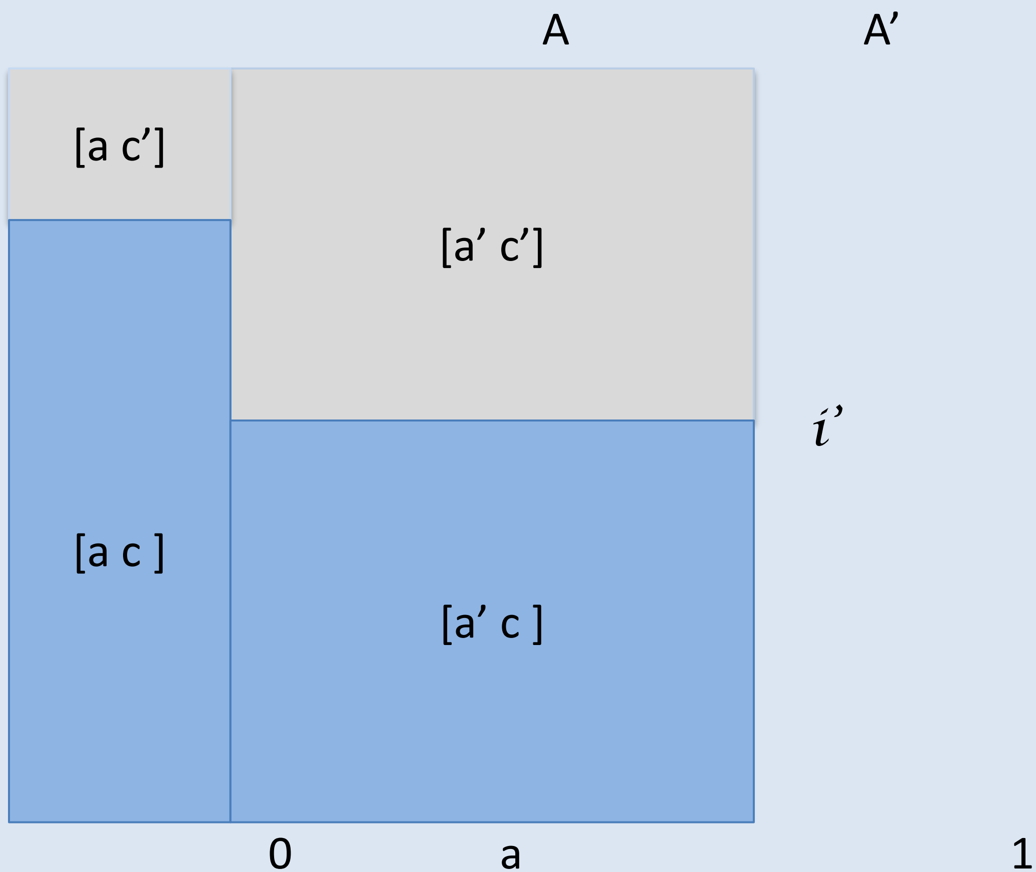
- A is the event "the left compartment occupies the whole tank";
  a, the capacity of A, is the degree to which the whole tank is occupied by A: $a = P(A)$

- C is the event "the tank is filled up with liquid";
  c (the volume of liquid in the tank) is the degree to which the tank is filled with liquid: $c = P(C)$

- The event *restricted* to the compartment A, "A is filled up with liquid" is a *conditional event* denoted by C/A;
  i (the level of liquid in A) is the degree to which A is filled with liquid: $i = P(C \mid A)$

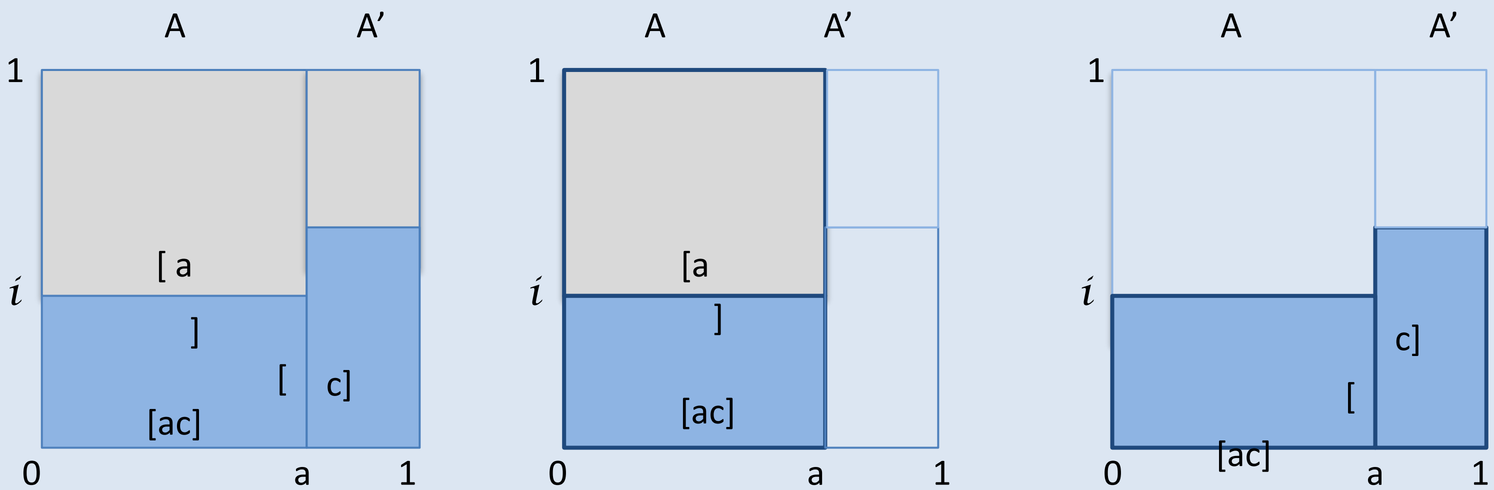
- A AND C is the event "the tank is occupied by A and C";
  [ac] (the volume of liquid common to A and C) is the joint probability of A and C, P(A AND C) which is measured by a x i: $P(A \text{ AND } C) = a \times i = P(A) \times P(C \mid A)$

The total probability rule



c = [ac] + [a’c] = (a x i) + (a’ x i’)

Bayes' rule



Two ways of measuring [ac]:

with respect to A:
[ac] is the fraction i of A

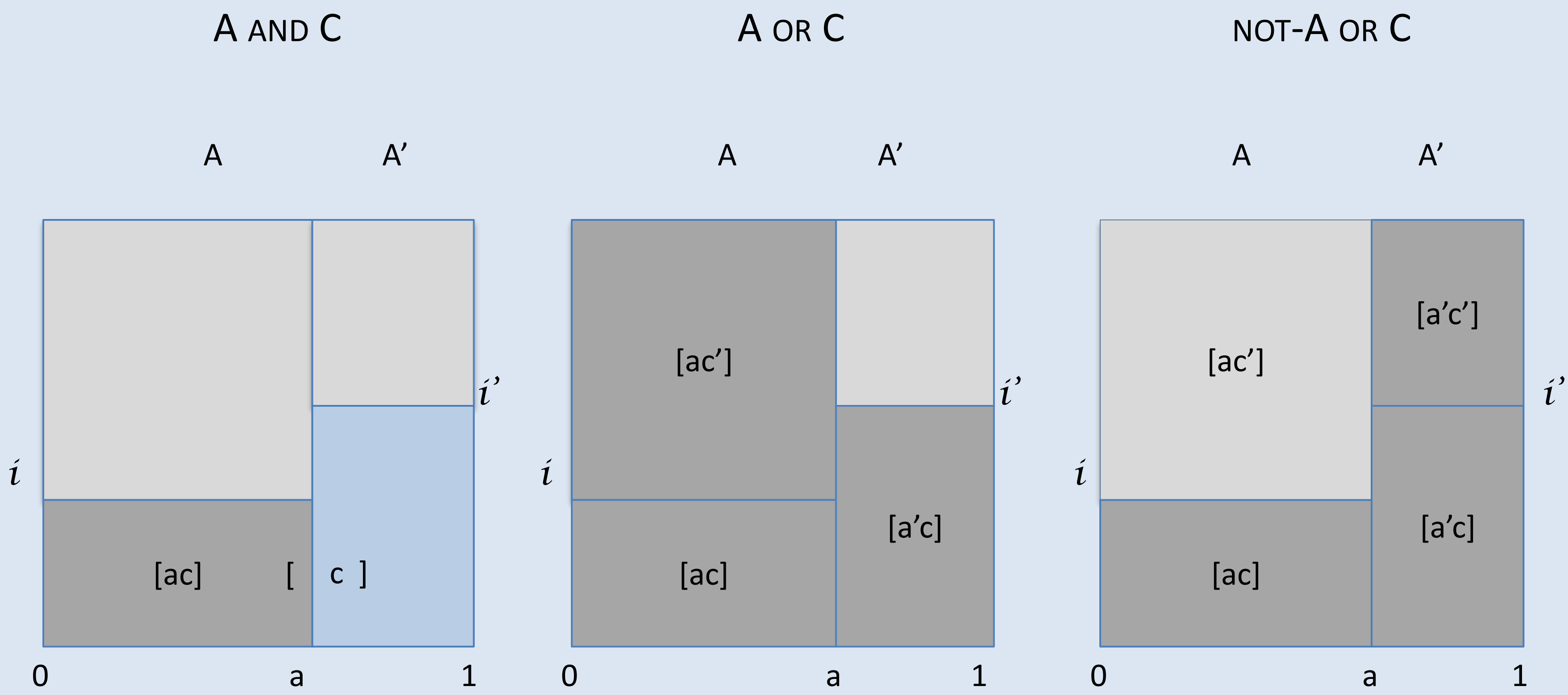
[ac] = a $\times$ i = a $\times$ P(C | A)

with respect to C:
[ac] is the fraction P(A | C) of C
[ac] = c $\times$ P(A | C)

hence:

a $\times$ P(C | A) = c $\times$ P(A | C)

Boolean operations



Interpreting probabilistic inferences

• ***Sentences*** (premises and conclusion) express **events represented by the constituents of the tank, viz. compartments, volumes or fillings**.

• ***Probabilities*** are represented by **the constituents' measures of capacity, amount, or level**.

Representation of sentences	
Qualitative (event)	Quantitative (probability)
constituents:	variables:
compartment A, A'…(event)	capacity a, a',…
volume C, C',…(event)	amount c, c',…
filling C|A, A|C,…(tri-event)	level i, i',…

Interpreting probabilistic inferences (continued)

What is an inference?

First, select one or several constituents of the device (such as A, A', C|A, C, etc.), materialising the premises and assume a fixed value for them.

Second, consider another constituent (the conclusion) and examine whether its value is constrained by the former constituents, that is, whether it is free to vary, and to what extent?

There are two cases:

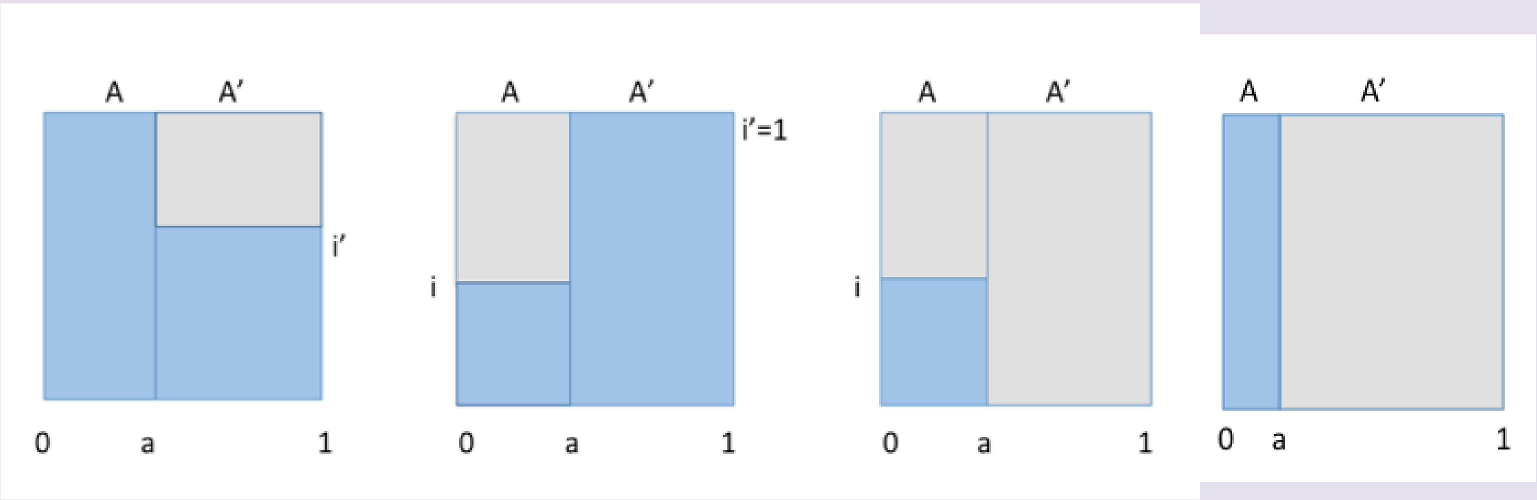
- The target constituent can take on any value: knowledge of the given constituents has no bearing on the target constituent, that is, the degree of belief in the premises does not determine that of the conclusion: the confidence in the conclusion can be anywhere in the interval [0, 1]. Normatively, this means that **one is permitted to have any degree of belief in the conclusion; in particular to have a low or a null degree of belief, however high the probability of the premise may be. There is no argument proper.**

- The target constituent can not take on any value because some values are **physically impossible:** knowledge of the given constituents constrains the target constituent, meaning that the degree of belief in the premises determines that of the conclusion: this defines an interval of confidence in the conclusion [l, u] (possibly a point value). Normatively, **one is not permitted to have a degree of belief in the conclusion out of the interval, but one may have any degree of belief inside the interval.**

I will now consider a few schemas, derive their coherence interval using the tank analogy, and then examine whether there are any circumstances in which a specific degree of belief in the conclusion is justifiable.

Contraposition: If A then C ∴ If not-C then not-A

The conclusion represents the degree to which the right compartment A’ contributes to the vacuum (the emptiness of the tank).



I II III IV

In the sure case (Fig.I) A is full (i = 1, full belief in the premise) all the empty volume is in A' so that i_{cp} = 1 and the contrapositive is fully believable.

In the general case the contribution of the right compartment A’ to the vacuum varies from zero when it is full (i_{cp} = 0, Fig. II) to a maximum when it is empty (Fig. III), yielding $i_{cp} = (1 - a) / (1 – c)$. But this maximum is susceptible of variation depending on the relative size of the compartments: as A decreases A' increases and the contribution of A' increases until i_{cp} reaches 1 (Fig. IV). In sum, the contrapositive can have all the degrees of confidence between 0 and 1:

$$\mathbf{i_{cp} = p(\text{if not-C then not-A}) \in [0,1]}$$

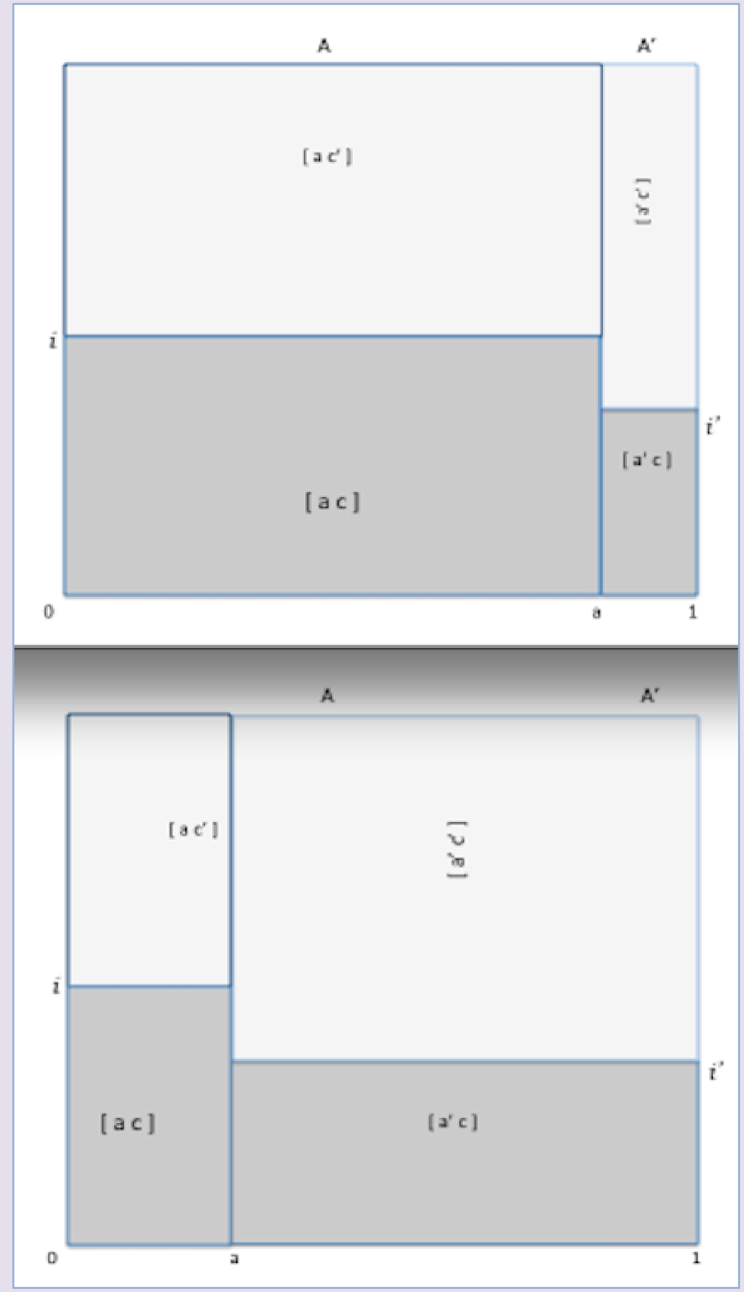
The contrapositive is often endorsed in common conversation, which is correct with a sure premise.

With an uncertain premise we have found that any assessment can be coherently given, that is, the reasoner should be agnostic about the conclusion. To analyse whether the endorsement of the conclusion can be justified, consider the case where $p(C|A)$ is just high enough to be assertable ($i \geq 1/2$).

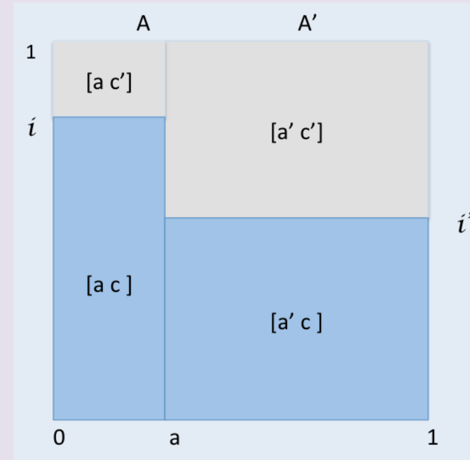
To assert *if not-C then not-A*, $p(\text{not-A}|\text{not-C})$ should be $\geq 1/2$.

When p(A) is large (top diagram), this cannot happen because whatever the value of i' the contribution of A' to the vacuum is too small (the ratio [a'c']/ c' is small), hence the contrapositive is not assertable.

On the contrary, when p(A) is small (bottom diagram), the contrapositive becomes highly assertable as the ratio [a'c']/ c' is large (unless i' is very high). High values for the contrapositive are even more easily reached if we assume that $p(C|A)$ is high.

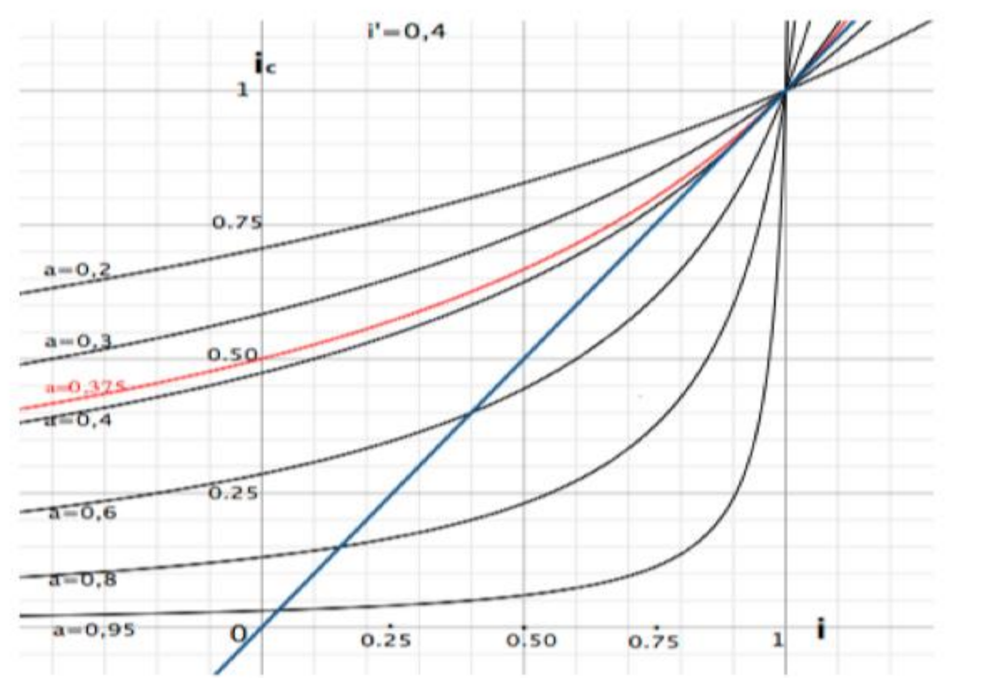


contraposition



Expressing i_{cp} as the ratio of the empty space in A',
[a’c’] = (1 - a)(1 - i')
to the whole empty space [ac’] + [a’c’] = 1-c:
$i_{cp} = (1 - a)(1 - i') / (1 - c)$
the equation of the contrapositive i_{cp} as a function of i = p(C│A) follows:
$i_{cp} = (1 - a)(1 - i') / [(1 - a)(1 - i') + a(1 - i)]$

The foregoing analysis can be verified with the equation and on the graph where the contrapositive i_{cp} is plotted as a function of i = p(C│A), with various values of a=p(A) as a parameter.



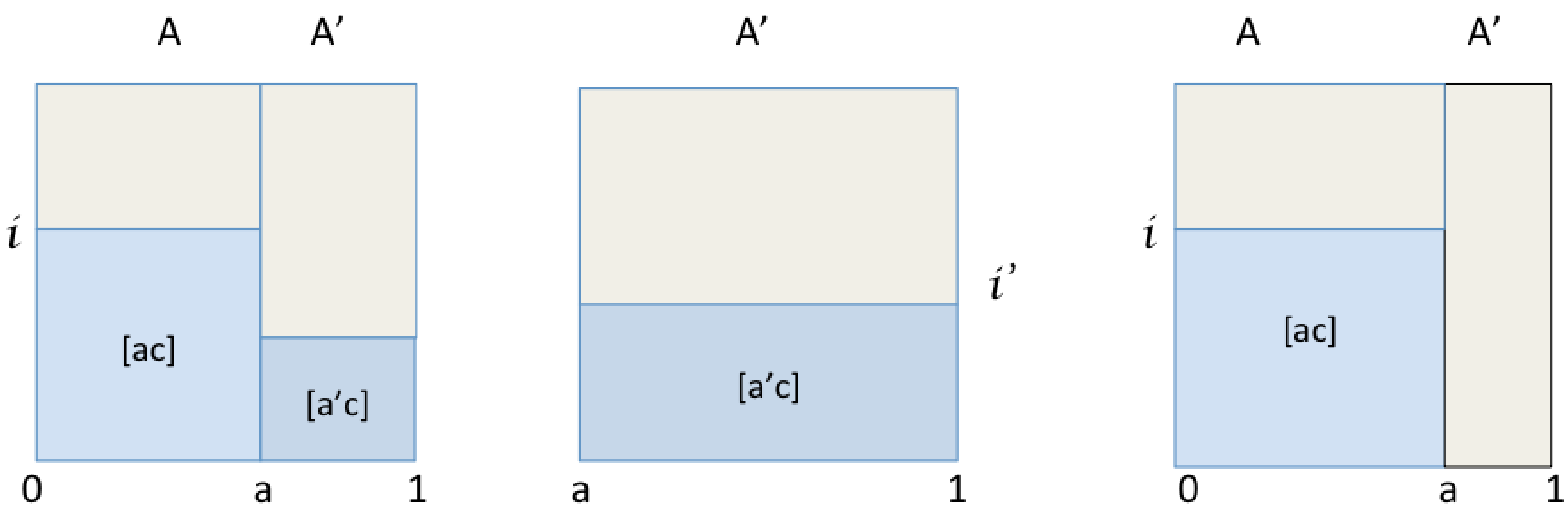
Probability of the contrapositive i_c plotted against i = P(C/A) for various values of a = P(A) and for i' = P(C/not A) fixed arbitrarily at 0.4. $i_c \geq i$ for $0 \leq i \leq j$ (j = intersect of f(i) and the diagonal).

Conversion: if A then C ∴ if C then A

The probability of the converse expresses the contribution of the left compartment to accommodate the liquid.

$i = p(\text{if A then C})$

$i_{cv} = p(\text{if C then A}) = [a\ c] / [a\ c] + [a\ c']$



$i_{cv} = [a\ c] / [a\ c] + [a'c]$

$i_{cv} = 0 / 0 + [a'c] = 0$

$i_{cv} = [a\ c] / [a\ c] + 0 = 1$

When A has a null capacity ($a = 0$) i_{cv} is null. When A is large enough to receive all the liquid it equals 1. All positions in between are possible, hence:

p(if C then A) = i_{cv} ∈ [0, 1]

Can conversion be a "reasonable" inference? (= rationally and pragmatically justified).
The inference is of practical interest if the premise is assertable ($i \geq 1/2$) and the inference is locally° conservative, so that the conclusion will be assertable too :

it is more likely than not that if A then C
therefore, it is more likely than not that if C then A

The inference will be even more valuable if it is locally robust.

it is almost certain that if A then C
therefore, it is almost certain that if C then A

The basic equation for conversion stems from Bayes's rule:

$p(C \mid A) = p(A \mid C) \times p(C) / p(A)$

$i_{cv} = a\, i / c = a\, i / (a\, i + (1 - a)\, i')$

so that looking for the conditions that satisfy the equation

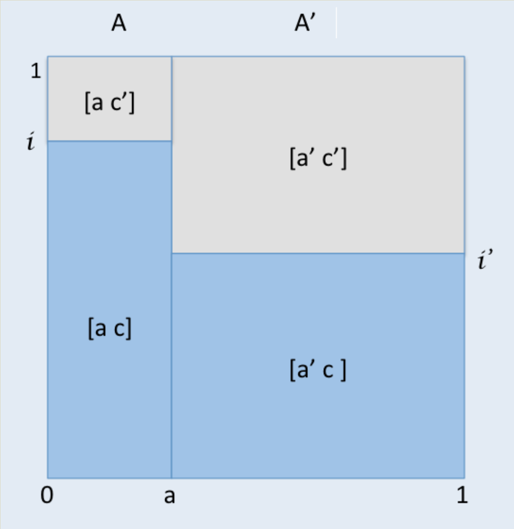
$i_{cv} \geq i$ means:

$a\, i / (a\, i + (1 - a)\, i') \geq i$, hence:

$a(1 - i) \geq (1 - a)\, i'$

i, the probability of the conditional is given;

a, the probability of its antecedent, and **i'** = p(if not-A then C) are parameters.



«° locally » = for some values of the parameter conversion

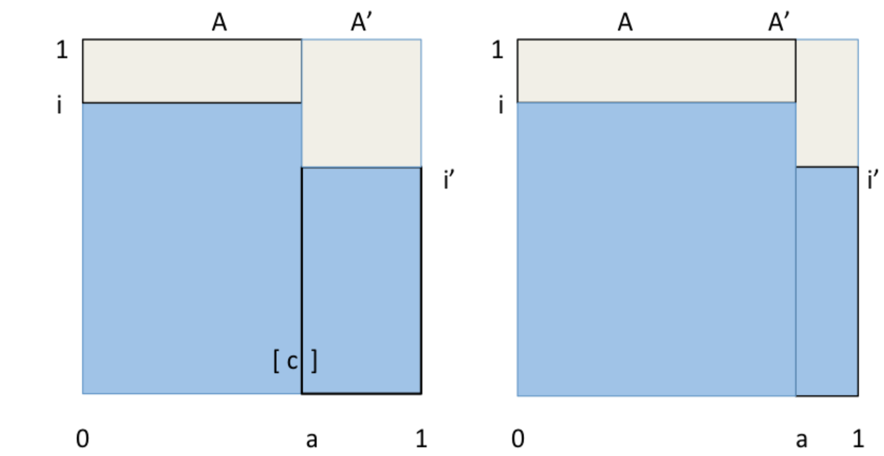
$a(1 - i) \geq (1 - a)\, i'$

The analogy shows that it is always possible to find a pair of values (a, i') that satisfies this equation because it amounts to asking whether it is possible for the area of the top left rectangle a(1 - i) to become greater than the area of the bottom right rectangle (1 - a) i'.

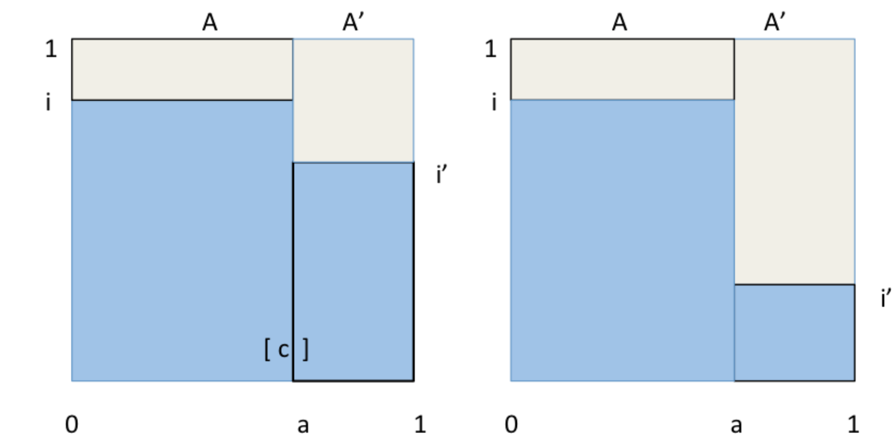
First, we assume $i \geq 1/2$, (and typically not far from 1).
We first fix **i'** arbitrarily and observe that to satisfy the equation it suffices to increase **a**, for all **i'**.

Reciprocally, we fix **a** arbitrarily and observe that to satisfy the equation it suffices to decrease **i'**.

In sum, conversion is «reasonable» in the sense <u>that the conclusion can be as probable as the premise provided that p(A) is high enough and that i' = p(if not-A then C) is not too high</u>.



i = p(if A then C)
i_{cv} = p(if C then A) = ai/c = ai/ai +(1-a)i'
$i_{cv} \geq i \Leftrightarrow a(1- i) \geq (1-a)i'$
increasing p(A) to increase i_{cv}= p(C|A)



i = p(if A then C)
i_{cv} = p(if C then A) = ai/c = ai/ai +(1-a)i'
$i_{cv} \geq i \Leftrightarrow a(1- i) \geq (1-a)i'$
decreasing i'= p(C|not-A) to increase i_{cv}= p(C|A)

conversion

$a(1 - i) \geq (1 - a)\, i'$

A numerical example:
believing rather strongly that *if A then C* ($i = 4/5$);
considering it as likely as not that *if not-A then C* ($i' = 1/2$)
with $a = 3/4$, the probability of *if C then A* is slightly greater than **i**
($i_{cv} \approx .83$): the conversion is highly reasonable.

Discussion:
Given a (high) degree of belief in a conditional, a (high) degree of belief in the converse can always be justified based on the assumption that a and i' are bounded (downward and upward, respectively).
This is the normative point of view.

But to ascertain that reasoners are coherent indeed, this assumption must be verified. Empirical data are necessary to know whether the reasoner makes the appropriate assessment of the additional premise.

One last point: assuming maximum uncertainty with regard to **a** and **i'** (both being as likely as not, $a = i' = 1/2$), and *if A then C* to be assertable (more likely than not, $i \geq 1/2$) then i_{cv} is also assertable ($i_{cv} \geq 1/2$) but lower than **i** (dissipative inference).

Conclusion

I have offered a normative (rational) justification for the relatively frequent endorsement of the conclusion of deductive schemas that, at first sight, seem useless because they are lower bounded by zero and probabilistically little informative or totally uninformative (when also upper bounded by 1).

The result is that there must always be an assumption of low belief or high belief, on the part of the reasoner, in an additional premise not explicitly present in the context.

This shortens the coherence interval and allows the reasoner, depending on the schema and the premise probability,
to assert the conclusion as more likely than not, or even
to have a strong belief in the conclusion.

Psychologically, there is a major question which I have not treated, but susceptible of being experimentally tested: are reasoners *actually* coherent when the additional premise is explicitly taken into account?

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